ADVANCED NUMBER THEORY FINAL EXAMINATION

This exam is of **50 marks** and is **3 hours long**. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly. Please sign the following statement:

I have not used any unfair or illegal means to answer any of the questions in this exam.

Name: **Signature:**

1. Prove there are infinitely many primes of the form $4n+1$. (5)

2. Prove that if $p \equiv 2 \bmod 3$ then the map $x \to x^3$ on $\mathbb{F}_p^*$ is an isomorphism. (5)

3a. Let f be a non-zero modular form of weight k for $SL_2(\mathbb{Z})$. For what weights k is f' a modular form? (5)

3b. Show that if f is a modular form of weight k then (10)

$$g(z) = \frac{1}{2\pi i} f'(z) - \frac{k}{12} E_2(z) f(z)$$

is a modular form of weight $k+2$, where $E_2(z)$ is the conditionally convergent Eisenstein series of weight 2 for $SL_2(\mathbb{Z})$. Recall that

$$z^{-2} E_2(\frac{-1}{z}) = E_2(z) + \frac{12}{2\pi i z}$$

3c. What is g when f is E_4? Your answer should be in terms of the generators of the algebra of modular forms given by the Eisenstein series E_4 and E_6. (5)

4. Count the number of projective solutions of the equation (5)

$$aX + bY + cZ = 0$$

in $\mathbb{P}^2(\mathbb{F}_p)$ where a, b and c are in $\mathbb{F}_p^*$.

5. Count the number of points in $\mathbb{P}^2(\mathbb{F}_p)$ on the hypersurface whose affine equation is given by

$$2X^3 + Y^3 = 1$$

5a. When $p = 29$ (5)

5b. When $p = 31$ (10)

You may want to use the following facts

- $31 = (-1+5\omega)(-1+5\omega^2)$ in $\mathbb{Z}[\omega]$ where $\omega^3 = 1$.
- $J(\chi, \chi) = a + b\omega$ where $a \equiv 2 \bmod 3$ and $b \equiv 0 \bmod 3$, $a^2 - ab + b^2 = 31$ and χ is a cubic character.
- $4^3 = 64$